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Type B c-Birkhoff polytopes are order polytopes

Esther Banaian*[1], Sunita Chepuri†[2], Emily Gunawan‡[3], and Jianping Pan§ [4]

[1] *Dept. of Mathematics, University of California, Riverside, Riverside, CA, USA*
[2] *Dept. of Mathematics and Computer Science, University of Puget Sound, Tacoma, WA, USA*
[3] *Dept. of Mathematics and Statistics, University of Massachusetts Lowell, Lowell, MA, USA*
[4] *School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ, USA*

Abstract. In a previous work, we defined (type A) c-Birkhoff polytopes and showed that they were unimodularly equivalent to order polytopes. In this extended abstract we answer the question: what about type B?

Keywords: Birkhoff polytope, Order polytope, Heap, Cambrian lattice, c-singleton

1 Introduction

Given a finite poset H, the order polytope $\mathcal{O}(H)$ is a well-understood polytope in $\mathbb{R}^{|H|}$ [11]. Its vertices are the indicator vectors of the order ideals of H, its dimension is $|H|$, and its normalized volume is the number of linear extensions of H.

On the other hand, let S_m denote the symmetric group on $[m] = \{1, \dots, m\}$. Given a permutation $w \in S_m$, let $X(w)$ be the corresponding permutation matrix, i.e., with 1's in row i and column $w(i)$ for all $i \in [m]$ and 0's everywhere else. The *Birkhoff polytope* for S_m is the convex hull of all permutation matrices [2].

In [4], Davis and Sagan studied the convex hull of 132 and 312 avoiding permutation matrices, a subpolytope of the Birkhoff polytope. They proved that the normalized volume of this polytope is the number of longest chains in the type A_{m-1} Tamari lattice. Inspired by their work and the fact that the 132 and 312 avoiding permutations are exactly the c-singletons for the Coxeter element $c = (12 \dots m)$ written in cycle notation, in [1] we defined a (type A) Birkhoff subpolytope $\mathsf{Birk}(c)$ to be the convex hull of permutation matrices corresponding to c-singletons for any Coxeter element c. We then proved that $\mathsf{Birk}(c)$ is integrally equivalent to the order polytope of the heap of the longest c-sorting word of S_m. A consequence of this result is that the normalized volume of $\mathsf{Birk}(c)$ is the number of longest (length $\binom{m}{2}$) chains in the (type A_{m-1}) c-Cambrian lattice [9].

In the present paper, we turn our attention to the Coxeter group B_n which is realized as the group of permutations v on $\pm[n] = \{-n, \dots, -1, 1, \dots, n\}$ satisfying $v(-k) =$

*esther.banaian@ucr.edu
†schepuri@pugetsound.edu
‡emily_gunawan@uml.edu
§jianping.pan@asu.edu

$-v(k)$; such permutations are called *signed permutations* on $[n]$. We can naturally embed each $v \in B_n$ into a permutation $\eta(v)$ in S_{2n} by identifying $-n, \ldots, -1, 1, \ldots, n$ with $1, \ldots, n, n+1, \ldots, 2n$, in this order.

For each Coxeter element c^B in B_n, we define a (type B) Birkhoff subpolytope.

Definition 1.1 (c^B-Birkhoff polytope)**.** Given a Coxeter element $c^B \in B_n$, let $\mathsf{Birk}(c^B)$ be the convex hull of

$$\{X(\eta(v)) \mid v \text{ is a } c^B\text{-singleton in } B_n\}.$$

The main goal of this paper is to give a proof sketch that $\mathsf{Birk}(c^B)$ is integrally equivalent to the order polytope of the heap of the longest c^B-sorting word of B_n (Theorem 3.13). As in the type A work [1], a consequence of this result is that the normalized volume of $\mathsf{Birk}(c^B)$ is the number of longest (length n^2) chains in the (type B_n) c^B-Cambrian lattice.

2 Background and notation

A *Coxeter system* (W, S) is a *Coxeter group* W together with a set S of generators for W called *simple reflections* subject to the relations $s^2 = e$ for all $s \in S$ and the braid relations $(st)^{m(s,t)} = e$ for all s, t such that $m(s,t) < \infty$. For $s, t \in S$ where $m(s,t) = 2$, we have $st = ts$, which we call a *commutation relation*. An application of a commutation relation to a product of simple reflections is called a *commutation move*. A *Coxeter element* c in W is a product of all simple reflections in any order, where each reflection appears exactly once.

Given $w \in W$, the minimum number of simple reflections among all expressions for w as a product of simple reflections is called the *length* of w, and is denoted by $\ell(w)$. A *reduced decomposition* of w is an expression $w = s_{i_1} \cdots s_{i_{\ell(w)}}$ realizing $\ell(w)$.

2.1 Type A_n and B_n permutations

This paper focuses on the Coxeter groups of type A and B, which we denote by A_n and B_n. We now review the combinatorial realizations of these groups in terms of permutations and signed permutations. For more details, see for example [3]. The simple reflections in A_n are denoted $s_1^A, \ldots, s_n^A$; and the simple reflections in B_n are denoted $s_0^B, s_1^B, \ldots, s_{n-1}^B$. We sometimes write s_k when W is understood.

Let A_n denote the symmetric group on $n+1$ elements. We can represent a permutation $w \in A_n$ in *one-line notation* as $w = w(1)w(2)\ldots w(n+1)$. The simple reflections for A_n are *adjacent transpositions* $s_k^A = (k \quad k+1)$ for $1 \leq k \leq n$. Distinct simple reflections satisfy the commutation relation $s_i^A s_j^A = s_j^A s_i^A$ if and only if $|i - j| > 1$. The longest element of A_n is the permutation $w_0^A = (n+1)n\ldots 321$ and $\ell(w_0^A) = \binom{n+1}{2}$.

Let B_n be the group of signed permutations on $\pm[n] = \{-n, \ldots, -2, -1, 1, 2, \ldots, n\}$ which satisfies $w(-k) = -w(k)$ for all $k \in [n]$. We write these permutations in *full one-line notation* as $w(-n)w(-n+1)\dots w(-1)w(1)w(2)\dots w(n)$ or in *window notation* as $w(1)w(2)\dots w(n)$. The simple reflections for B_n are $s_0^B = (-1 \quad 1)$ and $s_k^B = (-k-1 \quad -k)(k \quad k+1)$ for $k = 1, \dots, n-1$. As in A_n, distinct simple reflections in B_n satisfy the commutation relation $s_i^B s_j^B = s_j^B s_i^B$ if and only if $|i-j| > 1$. The longest element of B_n is the signed permutation $w_0^B = (-1)(-2)\dots(-n)$ in window notation and $\ell(w_0^B) = n^2$.

To simplify notation, we refer to a reduced decomposition $s_{i_1} \cdots s_{i_{\ell(w)}}$ of w in A_n or B_n via its *reduced word* $\left[i_1 \cdots i_{\ell(w)}\right]$. Given a reduced word $[u]$, the equivalence class consisting of all words that can be obtained from $[u]$ by a sequence of commutation moves is called the *commutation class* of $[u]$.

2.2 Heaps

We review the classical theory of heaps [13], following the exposition in [12].

Definition 2.1. Let W be the Coxeter group A_n or B_n. Given a reduced word $[a] = [a_1 \cdots a_\ell]$ of an element in W, consider the partial order $\preccurlyeq$ on the set $\{1, \dots, \ell\}$ obtained via the transitive closure of the relations

$$x \prec y$$

for $x < y$ such that $|a_x - a_y| \le 1$. For each $1 \le x \le \ell$, the *label* of the poset element x is a_x. This labeled poset is called the *heap* for $[a]$, denoted $\mathsf{Heap}([a])$.

The Hasse diagram for this poset with elements $\{1, \dots, \ell\}$ replaced by their labels is called the *heap diagram* for $[a]$. In our figures, we represent each label j by the simple reflection s_j for clarity.

Example 2.2.

1. The first two pictures in Figure 1 show the Hasse diagram and heap diagram of $\mathsf{Heap}([aaaa])$ for $[a] = [7145362]$. The elements of the underlying poset are $\{1, 2, \dots, 28\}$, and the possible labels are $\{1, 2, \dots, 7\}$.
2. The last two pictures in Figure 1 show the Hasse diagram and heap diagram of $\mathsf{Heap}([bbbb])$ for $[b] = [3012]$. The elements of the underlying poset are $\{1, 2, \dots, 16\}$, and the possible labels are $\{0, 1, 2, 3\}$.

We can understand the commutation class of $[a]$ by looking at linear extensions of $\mathsf{Heap}([a])$.

Definition 2.3. A *linear extension* $\pi = \pi(1) \cdots \pi(\ell)$ of a partial order $\preccurlyeq$ on $\{1, \dots, \ell\}$ is a total order on the poset elements that is consistent with the structure of the poset, that is, $x \prec y$ implies $\pi(x) < \pi(y)$. Given a reduced word $[a] = [a_1 \cdots a_\ell]$, a *labeled*

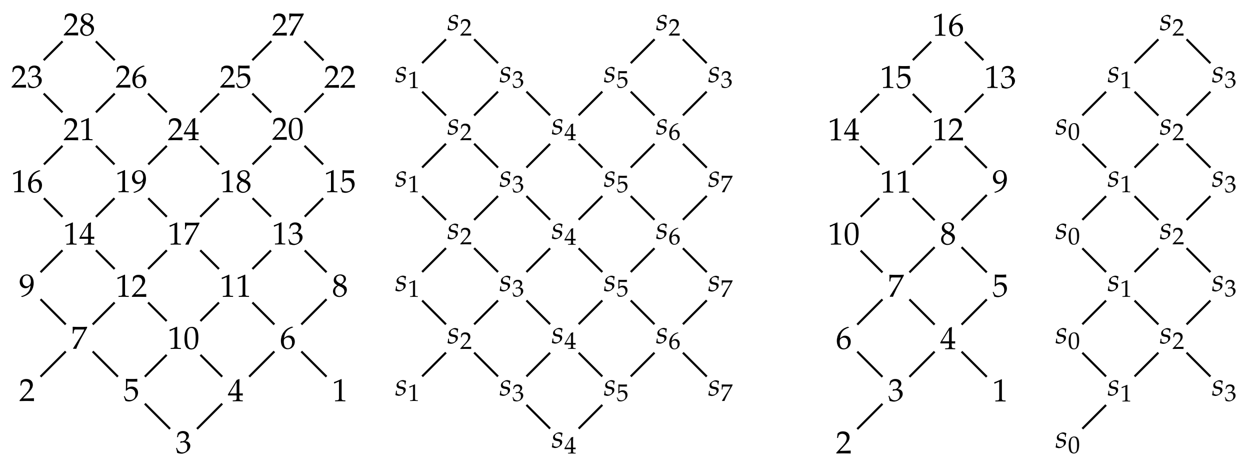


Figure 1: From left to right: Hasse diagram of the underlying poset of $\mathsf{Heap}([aaaa])$ for $[a] = [7145362]$ in A_7; heap diagram of $\mathsf{Heap}([aaaa])$; Hasse diagram of the underlying poset of $\mathsf{Heap}([bbbb])$ for $[b] = [3012]$ in B_4; heap diagram of $\mathsf{Heap}([bbbb])$

linear extension of $\mathsf{Heap}([a])$ is a word $\left[a_{\pi(1)} \cdots a_{\pi(\ell)}\right]$ where $\pi = \pi(1) \cdots \pi(\ell)$ is a linear extension of $\mathsf{Heap}([a])$.

Proposition 2.4. [12, Proof of Proposition 2.2] Given a reduced word $[a]$, the set of labeled linear extensions of the heap for $[a]$ is the commutation class of $[a]$.

2.3 The heap of the longest c-sorting word in A_n and B_n

The notion of c-sorting words was introduced by Reading in [10]. Fix a reduced word $[a_1 a_2 \ldots a_n]$ for a Coxeter element c, and define an infinite word

$$c^\infty := a_1 a_2 \ldots a_n \mid a_1 a_2 \ldots a_n \mid \cdots$$

The *c-sorting word* of $w \in W$ is the lexicographically first (as a sequence of positions in c^∞) subword of c^∞ that is a reduced word for w. Denote this word by $\mathsf{sort}_c(w)$. If a word $[u] = [u_1 \ldots u_\ell]$ is the c-sorting word of an element in W, we refer to $[u]$ as a *c-sorting word*.

In this paper, we are interested in the heap diagram of $\mathsf{sort}_c(w_0)$ for A_n and B_n. The following is proven in Sections 6.2 and 6.3 of [5], for types A and B, respectively.

Lemma 2.5. 1. The c-sorting word for w_0 in A_n is a concatenation of nonempty subwords of c, $\mathsf{sort}_c(w_0) = \left[K_1 \mid K_2 \mid \cdots \mid K_p\right]$ where $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_p$ as sets. For a construction of the heap diagram of $\mathsf{sort}_c(w_0)$ in type A, see for example [1, Algorithm 6.1].

2. The c-sorting word of w_0 in B_n is c^n, so, to draw the heap of c^n, we simply stack n "layers" of $\mathsf{Heap}(c)$.

See Figure 1 for the heap diagrams of $\mathsf{sort_c}(w_0)$ for $c = [7145362]$ in A_7 and for $c = [3012]$ in B_4.

2.4 c-singleton permutations

In [7], Hohlweg, Lange, and Thomas introduced the notion of c-singletons; following the survey [6], we will adopt the definition that $w \in W$ is a c-singleton if and only if some reduced word of w is a prefix of a word in the commutation class of $\mathsf{sort_c}(w_0)$. We will also use the following characterization of c-singletons, which follows from Proposition 2.4.

Lemma 2.6. An element $w \in W$ is a c-singleton if and only if there exists a reduced word $[u]$ of w and an order ideal I of $\mathsf{Heap}(\mathsf{sort_c}(w_0))$ such that $I = \mathsf{Heap}([u])$.

The c-singletons form a distributive sublattice of the right weak order on W, denoted $\mathcal{L}$(c-singletons) [7]. For a poset H, let $J(H)$ denote the lattice of order ideals of H. The following are due to [8, Proposition 3] and [1, Section 2].

Proposition 2.7. Let $[u] = \left[u_1 u_2 \ldots u_{\ell(w_0)}\right]$ denote $\mathsf{sort_c}(w_0)$, and consider the labeled poset $H = \mathsf{Heap}([u])$ on $\{1, 2, \ldots, \ell(w_0)\}$, following Definition 2.1. Given an order ideal I of H, let $[u]_I = [u_i]_{u \in I}$ denote the subword of $[u]$ at positions I.

1. The word $[u]_I$ is a c-sorting word.

2. The map

$$\begin{aligned} \mathrm{Perm}\colon J(H) &\to \mathcal{L}(\text{c-singletons}) \\ I &\mapsto [u]_I \end{aligned}$$

is a poset isomorphism, and the inverse map of Perm is

$$\begin{aligned} f : \mathcal{L}(\text{c-singletons}) &\to J(H) \\ w &\mapsto \mathsf{Heap}(\mathsf{sort_c}(w)) \end{aligned}$$

A *barring* of a set Z of integers is a partition of Z into two sets $\underline{Z}$ and $\overline{Z}$. If $d \in \underline{Z}$ (resp. $u \in \overline{Z}$), we call d a *lower-barred number* (resp. we call u a *upper-barred number*) and sometimes emphasize this by writing $\underline{d}$ (resp. $\overline{u}$).

Let c be a Coxeter element in W. Let $Z = [2, n] = \{2, 3, \ldots, n\}$ if $W = A_n$ and $Z = \pm[1, n-1] = \{-(n-1), \ldots, -2, -1, 1, 2, \ldots, n-1\}$ if $W = B_n$. First, the barring of $[2, n]$ and $[1, n-1]$ in types A and B respectively is defined as follows: if s_i appears after s_{i-1} in any reduced word of c, then i is a lower-barred number; otherwise i is an upper-barred number. For $W = B_n$, the barring of $[1, n-1]$ is extended to a barring of $Z = \pm[1, n-1]$ by specifying that the barring of $-i$ is opposite the barring of i.

We denote the lower-barred numbers by $d_1 < \ldots < d_r$ and the upper-barred numbers by $u_1 < \ldots < u_s$.

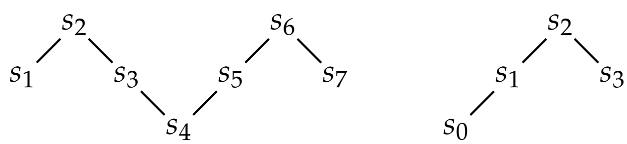
Remark 2.8 ([10, Section 3])**.** We can write a Coxeter element c of A_n as a single cycle of length $n+1$ of the form

$$c = (1\ \underline{d_1}\ \ldots\ \underline{d_r}\ (n+1)\ \overline{u_s}\ \ldots\ \overline{u_1})$$

and the Coxeter element c of B_n as a single cycle of length $2n$ of the form

$$c = (-n\ \underline{d_1}\ \ldots\ \underline{d_r}\ n\ \overline{u_s}\ \ldots\ \overline{u_1}).$$

Example 2.9. Consider the Coxeter elements $c^A = s_7^A s_1^A s_4^A s_5^A s_3^A s_6^A s_2^A$ in A_7 and $c^B = s_3^B s_0^B s_1^B s_2^B$ in B_4 as in Figure 1. Their heap diagrams are as follows.



Then $c^A = (1\ \underline{2}\ \underline{5}\ \underline{6}\ 8\ \overline{7}\ \overline{4}\ \overline{3}\ )$ and $c^B = (-4\ \underline{-3}\ \underline{1}\ \underline{2}\ 4\ \overline{3}\ \overline{-1}\ \overline{-2}\ )$.

3 Type B c-Birkhoff polytopes

For the rest of the paper, we will consider the Coxeter groups B_n and A_{2n-1}. The two groups are related by an "unfolding" injective homomorphism $\eta : B_n \to A_{2n-1}$, determined by

$$\eta(s_i^B) = \begin{cases} s_n^A & \text{if } i = 0 \\ s_{n+i}^A s_{n-i}^A & \text{if } i > 0 \end{cases}$$

for each simple reflection s_i^B of B_n.

Let c^B denote a Coxeter element of B_n and let $c^A = \eta(c^B)$; note that c^A is a Coxeter element of A_{2n-1}.

Remark 3.1. The homomorphism η can be alternatively defined as follows. Let $v \in B_n$, and let $w = \eta(v) \in A_{2n-1} = S_{2n}$. Then, for $k = 1, \ldots, n$, we have

$$\begin{aligned} &w(n+k) = v(k) + n \text{ and } w(n+1-k) = -v(k) + n + 1 && \text{if } v(k) > 0 \\ &w(n+k) = v(k) + n + 1 \text{ and } w(n+1-k) = -v(k) + n && \text{if } v(k) < 0 \end{aligned}$$

For example, let $v = (-1)\,(-4)\,3\,(-2) \in B_4$ in window notation, and let $w = \eta(v)$; then $w = 6\,2\,8\,5\,4\,1\,7\,3$. A second example is $\eta(c^B) = c^A$, where c^A and c^B are as in Example 2.9; the symmetry we see in the barring of $[2,7]$ in this example holds in general:

Lemma 3.2. In the barring of $[2, 2n-1]$ associated to the Coxeter element $\eta(c^B) \in A_{2n-1}$, the integer i is lower-barred if and only if $2n+1-i$ is upper-barred.

Define the type B_n c^B-Birkhoff polytope, denoted $\mathsf{Birk}(c^B)$, to be the convex hull of

$$\{X(\eta(v)) \mid v \in B_n \text{ is a } c^B\text{-singleton}\}.$$

The vertices of $\mathsf{Birk}(c^B)$ are precisely the permutation matrices $X(\eta(v))$ where v is a c^B-singleton. Let $\mathsf{Aff}(c^A)$ (resp. $\mathsf{Aff}(c^B)$) denote the affine hull of the vertices of $\mathsf{Birk}(c^A)$ (resp. $\mathsf{Birk}(c^B)$).

3.1 Reflection-invariant order ideals and rotation-invariant matrices

Let $H^B = \mathsf{Heap}(\mathrm{sort}_{c^B}(w_0^B))$ and $H^A = \mathsf{Heap}(\mathrm{sort}_{c^A}(w_0^A))$.

Lemma 3.3. H^A has reflectional symmetry about the vertical y-axis; the right side of H^A has the same underlying poset as H^B.

See Figure 1 for the heap diagrams of H^A and H^B where $c^B = s_3^B s_0^B s_1^B s_2^B$and $c^A = \eta(c^B) = s_7^A s_1^A s_4^A s_5^A s_3^A s_6^A s_2^A$. This example illustrates Lemma 3.3.

Let $\rho : H^A \to H^A$ be the “reflection" map which sends every vertex in the heap diagram of H^A to its reflection about the y-axis. This induces a map $J(H^A) \to J(H^A)$ which we will also call ρ. We say that $I \in J(H^A)$ is *reflection-invariant* if $\rho(I) = I$. Let $J(H^A)^F$ denote the set of reflection-invariant order ideals in $J(H^A)$.

By Lemma 3.3 we can view H^B as the subposet on the right side of H^A. This allows us to define a bijection $\alpha : J(H^B) \to J(H^A)^F$ as follows: if I^v is an order ideal of H^B, let $\alpha(I^v)$ be the order ideal of H^A that contains elements on the right side of H^A corresponding to the elements of I^v as well as the image of these elements under ρ. The inverse map $\beta : J(H^A)^F \to J(H^B)$ ignores the elements of $I^w \in J(H^A)^F$ labeled $1, \ldots, n-1$ and identifies the rest with the corresponding elements in H^B.

Consider the poset isomorphisms, Perm and f, defined in Proposition 2.7. To distinguish between type A and type B versions of these maps, we will sometimes use a superscript A or B.

Lemma 3.4. 1. If v is a c^B-singleton, then $\eta(v)$ is a c^A-singleton.

2. If $I^w \in J(H^A)$ is reflection-invariant and $w = \mathrm{Perm}^A(I^w)$, then $w = \eta(v)$ for some c^B-singleton v. Conversely, if v is a c^B-singleton, then $f^A(\eta(v))$ is reflection-invariant.

Proof. (1) Let v be a c^B-singleton and let I^v denote the order ideal $f^B(v)$ of H^B. Define I^w to be the order ideal $\alpha(I^v)$ of H^A. Then by Proposition 2.7 there is a c^A-singleton w where $w = \text{Perm}^A(I^w)$. By construction, $w = \eta(v)$.

(2) Suppose I^w is a reflection-invariant order ideal of H^A. Define I^v to be the order ideal $\beta(I^w)$ of H^B. Then by Proposition 2.7 there is a c^B-singleton v where $v = \text{Perm}^B(I^v)$. By construction, $w = \eta(v)$.

Conversely, if v is a c^B-singleton, then $v = \text{Perm}(I^v)$ for some $I^v \in J(H^B)$. By construction $\eta(v) = \text{Perm}^A(\alpha(I^v))$. Since $\alpha(I^v)$ is reflection-invariant and since f^A and Perm^A are inverse maps, the claim follows. □

It follows from Lemma 3.4(1) that $\mathsf{Birk}(c^B)$ is a subpolytope of the c^A-Birkhoff polytope $\mathsf{Birk}(c^A)$ and that $\mathsf{Aff}(c^B)$ is an affine subspace of $\mathsf{Aff}(c^A)$.

Definition 3.5. Let $w \in A_{m-1} = S_m$. The *reverse* of w, denoted w^{rev}, is the result of writing w in one-line notation backwards; that is, $w^{\text{rev}}(i) = w(m+1-i)$. The *complement* of w, denoted w^{comp}, is the result of replacing every entry i in the one-line notation of w with $m+1-i$. The *reverse-complement* of w, denoted w^{revcomp}, is the result of taking the complement of the reverse of w; that is, $w^{\text{revcomp}}(i) = m+1-w(m+1-i)$.

The permutation matrix $X(w^{\text{rev}})$ is the result of reflecting the permutation matrix $X(w)$ with respect to a horizontal line, while $X(w^{\text{comp}})$ is the result of reflecting $X(w)$ with respect to a vertical line. The composition of these two actions is the 180 degree rotation, and thus the 180 degree rotation of the permutation matrix $X(w)$ is the permutation matrix $X(w^{\text{revcomp}})$ of the reverse-complement of w.

As a consequence, given a permutation $w \in S_{2n}$, its permutation matrix $X(w)$ is invariant under 180 degree rotation if and only if w is equal to its reverse-complement, that is, for all $1 \le i \le 2n$, $w(i) = w^{\text{rev}}(i) = 2n+1-w(2n+1-i)$. Equivalently, $X(w)$ is invariant under 180 degree rotation if and only if w satisfies

$$w(n+k) + w(n+1-k) = 2n+1 \tag{3.1}$$

for all $k = 1, \dots, n$.

Let A^{180}_{2n-1} denote $\{w \in A_{2n-1} \mid w \text{ satisfies } (3.1)\}$, that is, A^{180}_{2n-1} is the set of permutations in A_{2n-1} whose permutation matrices are invariant under 180 degree rotation. Remark 3.1 implies the following.

Lemma 3.6. The image $\eta(B_n)$ is equal to A^{180}_{2n-1}.

3.2 Type B lattice-preserving projection

In [1, Section 5], we defined a projection Π_{c^A} from the space of $(2n) \times (2n)$ $\mathbb{R}$-valued matrices to $\mathbb{R}^{\binom{2n}{2}}$ by choosing exactly $\binom{2n}{2}$ positions from a matrix X; the positions are determined by the Coxeter element c^A. We proved in [1, Theorem 5.9] that Π_{c^A} is injective on $\mathsf{Aff}(c^A)$.

Proposition 3.7 (Zero relations [1, Proposition 4.4])**.** If $X \in \mathsf{Aff}(c^A)$, then X satisfies the following.

- For each upper-barred u, we have $X(i,u) = 0$ for all $1 \le i \le \min(u-1, n+1-u)$.
- For each lower-barred d, $X(i,d) = 0$ for all $\max(d+1, n+3-d) \le i \le n+1$.

The projection Π_{c^A} never includes positions on the main diagonal or positions whose entries are guaranteed to be zero for $X \in \mathsf{Aff}(c^A)$ by Proposition 3.7. When positions below the main diagonal are included, these positions must come from the bottom half of X.

In the following, we view the projection Π_{c^A} as a subset of $[2n] \times [2n]$.

Lemma 3.8. The map Π_{c^A} chooses n^2 positions from the top half (rows 1 through n) of a matrix X.

Proof. Let $X \in \mathsf{Aff}(c^A)$. The subset of entries taken by Π_{c^A} in the top half of X are exactly those above the main diagonal and not in a spot which is guaranteed to be 0 (as described in Proposition 3.7). The number of entries above the main diagonal and in the top half are $(2n-1) + (2n-2) + \cdots + n = n^2 + \binom{n}{2}$. The total number of entries in X guaranteed to be 0 is $2\binom{n}{2}$, and Lemma 3.2 implies that exactly half of these will be above the main diagonal. Therefore, Π_{c^A} restricted to $[n] \times [2n]$ has exactly n^2 entries. □

It follows from Lemma 3.6 that, for each $v \in B_n$, the top half of $X(\eta(v))$ determines its bottom half. In view of this fact and Lemma 3.8, we define

$$\Pi_{c^B} : \{(2n) \times (2n) \ \mathbb{R}\text{-valued matrices}\} \to \mathbb{R}^{n^2}$$

to be the result of restricting Π_{c^A} to $[n] \times [2n]$.

Example 3.9. Consider $c^A = (1\ \underline{2}\ \underline{5}\ \underline{6}\ 8\ \overline{7}\ \overline{4}\ \overline{3}\ )$ and $c^B = (-4\ \underline{-3}\ \underline{1}\ \underline{2}\ 4\ \overline{3}\ \overline{-1}\ \overline{-2}\ )$ from Example 2.9. Figure 2 shows the projection Π_{c^A} and Π_{c^B} (first two pictures). It also shows the permutation matrix for $\eta(v) \in S_8$ for the c^B-singleton $v = (1\,4\,3\ -1\ -4\ -3)(2\quad -2)$ with circles around entries recorded by Π_{c^B} (third picture).

Proposition 3.10. The map Π_{c^B} is a linear transformation which is injective on $\mathsf{Aff}(c^B)$ and sends integral points to integral points.

Proof. Consider $X \in \mathsf{Aff}(c^B)$. The first k positions in row k for $1 \le k \le n$ are not guaranteed to be zero by Proposition 3.7 and also not chosen by Π_{c^B}. The other entries in the row are either guaranteed to be zero by Proposition 3.7 or are chosen by Π_{c^B}.

	28	X	X	24	19	X	12
		X	X	25	20	6	13
			X	26	21	7	14
				27	22	8	15
					23	9	16
	3			X		10	17
4	1			X	X		18
2	X	5	11	X	X		

	16	X	X	12	8	X	4
		X	X	13	9	1	5
			X	14	10	2	6
				15	11	3	7
				X			
				X	X		
	X			X	X		

0	①	0	0	⓪	⓪	0	⓪
0	0	0	0	①	⓪	1	⓪
0	0	0	0	⓪	①	⓪	⓪
1	0	0	0	⓪	⓪	⓪	⓪
0	0	0	0	0	0	0	1
0	0	1	0	0	0	0	0
0	0	0	1	0	0	0	0
0	0	0	0	0	0	1	0

Figure 2: First two pictures: Projections Π_{c^A} and Π_{c^B} of Example 3.9. Red X's indicate the entries which must be zero by Proposition 3.7. Numbers indicate entries chosen by Π_{c^A} and Π_{c^B}, respectively, in order. Third picture: The permutation matrix for $\eta(v) \in S_8$ for $v = (1\ \ 4\ \ 3\ \ -1\ \ -4\ \ -3)(2\ \ -2)$, circling the entries recorded by Π_{c^B}.

Since $X \in \mathsf{Aff}(c^B)$, X is also in $\mathsf{Aff}(c^A)$. Thus, [1, Theorem 4.11 "Top sum relations"] tells us that there are k relations involving entries of rows 1 through k. Working from $k = 1$ to $k = n$, we can use these relations, the values of the entries in rows above row k, and the values of the entries in positions $k+1$ through $2n$ of row k to determine the first k entries of the kth row (see the proof of [1, Theorem 5.9]). This shows that Π_{c^B} determines the entire top half of X. By Lemma 3.6, we know that X is invariant under 180 degree rotation, and thus the top half of X determines the bottom half of X. □

3.3 Proof sketch of main theorem

In this section, we will prove our main theorem using a composition of several maps. The maps we will define and use throughout the section are depicted in the following commutative diagram.

$$\begin{array}{ccccc} \mathsf{Aff}(c^A) & \xrightarrow{\Pi_{c^A}} & \Pi_{c^A}(\mathsf{Aff}(c^A)) & \xrightarrow{\mathcal{U}_{c^A}} & \mathbb{R}^{\binom{2n}{2}} \\ \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle L} & & \downarrow{\scriptstyle P} \\ \mathsf{Aff}(c^B) & \xrightarrow{\Pi_{c^B}} & \Pi_{c^B}(\mathsf{Aff}(c^B)) & \xrightarrow{\mathcal{U}_{c^B}} & \mathbb{R}^{n^2} \end{array} \tag{3.2}$$

We first define the map L. Since Π_{c^B} is injective, if we restrict its codomain to be $\Pi_{c^B}(\mathsf{Aff}(c^B))$ then it is bijective. Thus it has an inverse $\Pi_{c^B}^{-1} : \Pi_{c^B}(\mathsf{Aff}(c^B)) \to \mathsf{Aff}(c^B)$. Define $L = \Pi_{c^A} \circ \iota \circ \Pi_{c^B}^{-1}$. Since L is a composition of injective functions, L is injective.

Recall the map f from Proposition 2.7 and let $o(I)$ denote the indicator vector of an order ideal I of a poset. In the proof of our main theorem in [1], we showed the existence of a unimodular transformation $\mathcal{U}_{c^A}$ such that, for all c^A-singletons w,

$(\mathcal{U}_{c^A} \circ \Pi_{c^A})(X(w)) = o(f^A(w))$. Since L maps from $\Pi_{c^B}(\mathsf{Aff}(c^B))$ to $\Pi_{c^A}(\mathsf{Aff}(c^A))$, we can consider the composition $\mathcal{U}_{c^A} \circ L$.

Let $\mathsf{Aff}(J(H^A))$ denote the affine hull of indicator vectors of order ideals of H^A and define $\mathsf{Aff}(J(H^A)^F)$ accordingly.

Lemma 3.11. The image $(\mathcal{U}_{c^A} \circ L)(\Pi_{c^B}(\mathsf{Aff}(c^B)))$ is equal to $\mathsf{Aff}(J(H^A)^F)$.

Proof. Given a c^A-singleton w, from [1, Theorem 6.21], $\mathcal{U}_{c^A} \circ \Pi_{c^A}(X(w)) = o(f^A(w))$. Given a c^B-singleton v, we know from Lemma 3.4(2) that the order ideal $f^A(\eta(v))$ is reflection-invariant. Therefore, applying $\mathcal{U}_{c^A} \circ L \circ \Pi_{c^B} = \mathcal{U}_{c^A} \circ \Pi_{c^A} \circ \iota$ to the set

$$\{X(\eta(v)) \mid v \text{ is a } c^B\text{-singleton}\}$$

produces the set of indicator vectors of the order ideals in $J(H^A)^F$, and the claim follows. □

Now we will define the map P. In [1, Section 6.1], we defined a specific linear extension, π_A, of H^A coming from the construction of the "diagonal reading word". There is an induced linear extension on H^B, π_B, viewing H^B as a subposet of H^A. Let $P : \mathbb{R}^{\binom{2n}{2}} \to \mathbb{R}^{n^2}$ be the linear map defined by $P(\mathbf{e}_i) = \mathbf{0}$ if $\pi_A^{-1}(i)$ is labeled $k < n$ and otherwise $P(\mathbf{e}_i) = \mathbf{e}_j$ if $\pi_A^{-1}(i)$ and $\pi_B^{-1}(j)$ are associated to the same poset element, conflating H^B with the right side of H^A. From this description, we see P is full-rank and lattice-preserving.

Example 3.12. Recall that Figure 1 showed H^A for $c = [7145362]$ in A_7 and H^B for $c = [3012]$ in B_4. The linear extension π_A is given by the following permutation in two-line notation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\ 11 & 1 & 2 & 6 & 3 & 12 & 4 & 18 & 5 & 7 & 13 & 8 & 19 & 9 & 24 & 10 & 14 & 20 & 15 & 25 & 16 & 27 & 17 & 21 & 26 & 22 & 28 & 23 \end{pmatrix}$$

This induces the following linear extension π_B:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 4 & 1 & 2 & 5 & 8 & 3 & 6 & 9 & 12 & 7 & 10 & 13 & 15 & 11 & 14 & 16 \end{pmatrix}$$

We can use these maps to compute $P : \mathbb{R}^{28} \to \mathbb{R}^{16}$. As an example, we will compute $P(\mathbf{e}_1)$ and $P(\mathbf{e}_2)$. Since $\pi_A^{-1}(1) = 2$ and 2 has label $1 < 4 = n$, $P(\mathbf{e}_1) = \mathbf{0}$. To compute $P(\mathbf{e}_2)$, first notice that $\pi_A^{-1}(2) = 3$ and 3 has label $4 \not< 4$. Since the element 3 in H^A corresponds to the element 2 in H^B and $\pi_B^{-1}(1) = 2$, $P(\mathbf{e}_2) = \mathbf{e}_1$.

Define $\mathcal{U}_{c^B} := P \circ \mathcal{U}_{c^A} \circ L$.

Theorem 3.13. The map $\mathcal{U}_{c^B} \circ \Pi_{c^B}$ is a unimodular transformation such that, for all vertices $X(\eta(v))$ of $\mathsf{Birk}(c^B)$, we have $(\mathcal{U}_{c^B} \circ \Pi_{c^B})(X(\eta(v))) = o(f^B(v))$. In particular, $\mathsf{Birk}(c^B)$ is integrally equivalent to $\mathcal{O}(H^B)$.

Proof. From the commutative diagram (3.2), we see that $\mathcal{U}_{c^B} \circ \Pi_{c^B} = P \circ \mathcal{U}_{c^A} \circ \Pi_{c^A} \circ \iota$. Thus, $(\mathcal{U}_{c^B} \circ \Pi_{c^B})(X(\eta(v))) = P(o(f^A(\eta(v))))$. By the definition of P, we conclude $P(o(f^A(\eta(v)))) = o(f^B(v))$, as desired.

The maps L and $\mathcal{U}_{c^A}$ are injective. Since P preserves information from the right side of H^A and kills information from the left side of H^A, P is injective on $\mathsf{Aff}(J(H^A)^F)$. The image of $\mathcal{U}_{c^A} \circ L$ is $\mathsf{Aff}(J(H^A)^F)$ by Lemma 3.11, so $P \circ \mathcal{U}_{c^A} \circ L = \mathcal{U}_{c^B}$ is injective.

Since Π_{c^B} is injective on $\mathsf{Aff}(c^B)$, the composition $\mathcal{U}_{c^B} \circ \Pi_{c^B}$ is also injective on $\mathsf{Aff}(c^B)$. Furthermore, since P, $\mathcal{U}_{c^A}$, Π_{c^A}, and ι are lattice-preserving, we have that $\mathcal{U}_{c^B} \circ \Pi_{c^B}$ is also lattice-preserving. Therefore, $\mathcal{U}_{c^B} \circ \Pi_{c^B}$ is a unimodular transformation. □

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