Lecture 5: PRG *Construction (Cont.)*

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> ***Warning:*** This document is a rough draft, so it may contain bugs. Please feel free to email me with corrections.

Recap

Last lecture we started to show how to construct a PRG from any OWP f.

1. We first showed that it suffices to construct a PRG with a single bit stretch, i.e. a PRG

$$G : \{0,1\}^\lambda \to \{0,1\}^{\lambda+1}.$$

   This is the case since we showed that we can stretch any such PRG into one that has a polynomial stretch $k = \text{poly}(\lambda)$ by concatenating G k times

$$G^k : \{0,1\}^\lambda \to \{0,1\}^{\lambda+k}.$$

2. We then defined the notion of a *hardcore predicate*

   Definition 1. $P : \{0,1\}^\lambda \to \{0,1\}$ is a hardcore predicate of f if for every poly-size $\mathcal{A}$ there exists a negligible function μ such that for every $\lambda \in \mathbb{N}$,

$$\Pr_{x \leftarrow \{0,1\}^\lambda}[\mathcal{A}(f(x)) = P(x)] \leq \frac{1}{2} + \mu(\lambda).$$

3. We showed that if the OWP f had a hardcore predict $P : \{0,1\}^\lambda \to \{0,1\}$ then the following function is a PRG:

$$G(x) = f(x) \circ P(x).$$

Thus, it remains to argue that every OWP has a hardcore predicate. Goldreich and Levin proved that every one-way function has a randomized hardcore predicate.

Goldreich-Levin Theorem

Theorem 2. *[1] If f is a one-way function, then the following randomized predicate*

$$P(x,r) := x \cdot r \mod 2 = \sum_{i \in \lambda} x_i r_i \mod 2$$

is a hardcore predicate for f. Namely, for every poly-size $\mathcal{A}$ there exists a negligible function $\mu : \mathbb{N} \to [0,1]$ such that for every $\lambda \in \mathbb{N}$

$$\Pr_{U_\lambda \leftarrow \{0,1\}^\lambda}[\mathcal{A}(f(U_\lambda), r) = P(U_\lambda, r)] \leq \frac{1}{2} + \mu(\lambda)$$

This theorem implies that

$$G(x,r) = f(x) \circ r \circ P(x,r)$$

is a PRG. In this lecture our focus is on proving Theorem 2.

Proof of Theorem 2 Suppose for contradiction that there exists a poly-size adversary $\mathcal{A}$ and a non-negligible function $\epsilon : \mathbb{N} \to [0,1]$ such that for every $\lambda \in \mathbb{N}$,

$$\Pr_{U_\lambda \leftarrow \{0,1\}^\lambda}[\mathcal{A}(f(U_\lambda), r) = P(U_\lambda, r))] \geq \frac{1}{2} + \epsilon(\lambda)$$

We will show how to use $\mathcal{A}$ to construct an adversary $\mathcal{B}$ that breaks the one-wayness of f.

Since $\mathcal{A}$ is only assumed to predict with small advantage ϵ, we cannot expect $\mathcal{B}$ to invert every given image $y = f(x)$. Indeed, it may be that $\mathcal{A}$ always fails when given an input whose first λ-bits is y. So which y's can we hope to invert?

Define

$$\mathsf{GOOD} = \{x \in \{0,1\}^\lambda : \Pr[\mathcal{A}(f(x), r) = x \cdot r] \geq \frac{1}{2} + \frac{\epsilon(\lambda)}{2}$$

By Markov's inequality

$$\Pr_{x \leftarrow U_\lambda}[x \in \mathsf{GOOD}] \geq \frac{\epsilon(\lambda)}{2}$$

To expand on this, denoting by

$$p = \Pr_{x \leftarrow U_\lambda}[x \in \mathsf{GOOD}],$$

it holds that

$$\frac{1}{2} + \epsilon(\lambda) \leq$$
$$\Pr_{x \leftarrow U_\lambda}[\mathcal{A}(f(x), r) = P(x, r)] =$$
$$\Pr_{x \leftarrow \mathsf{GOOD}}[\mathcal{A}(f(x), r) = P(x, r)] \cdot \Pr_{x \leftarrow U_\lambda}[x \in \mathsf{GOOD}] + \Pr_{x \leftarrow \neg\mathsf{GOOD}}[\mathcal{A}(f(x), r) = P(x, r)] \cdot \Pr_{x \leftarrow U_\lambda}[x \notin \mathsf{GOOD}] =$$
$$\Pr_{x \leftarrow \mathsf{GOOD}}[\mathcal{A}(f(x), r) = P(x, r)] \cdot p + \Pr_{x \leftarrow \neg\mathsf{GOOD}}[\mathcal{A}(f(x), r) = P(x, r)] \cdot (1 - p) \leq$$
$$p + \frac{1}{2} + \frac{\epsilon(\lambda)}{2},$$

which implies that indeed $p \geq \frac{\epsilon(\lambda)}{2}$.

We will construct an algorithm $\mathcal{B}$ that succeeds in inverting every $f(x)$ for $x \in \mathsf{GOOD}$ with non-negligible probability.

Special case 1: $\mathcal{A}$ predicts $P(x, r)$ perfectly for every $x \in \mathsf{GOOD}$

As a warmup, suppose that $\mathcal{A}$ is a perfect predictor that for every $x \in \mathsf{GOOD}$ and every $r \in \{0,1\}^\lambda$

$$\mathcal{A}(f(x), r) = x \cdot r$$

With such a perfect predictor it is easy to invert the OWF f. Specifically, the inverter $\mathcal{B}$, given $y = f(x)$ where $x \leftarrow U_\lambda$, does the following:

1. For every $i \in [\lambda]$ compute $x'_i = \mathcal{A}(y, e_i)$.
2. Output $x = (x'_1, \ldots, x'_\lambda)$.

By our assumption that $\mathcal{A}$ is a perfect predictor, it follows that

$$x'_i = x \cdot e_i = x_i,$$

as desired.

Unfortunately, our predictor $\mathcal{A}$ may not be perfect. Next, we relax the perfect condition as follows.

Special case 2: There exists a non-negligible $\epsilon = \epsilon(\lambda)$ such that for every $x \in \mathsf{GOOD}$, $\mathcal{A}$ predicts $P(x, r)$ with probability $3/4 + \epsilon$

We can use this predictor to construct an algorithm $\mathcal{B}$ that for every $x \in \mathsf{GOOD}$, given $f(x)$ finds an inverse with overwhelming probability (i.e., probability $1 - \mu$ for some negligible function μ). The inverter $\mathcal{B}$, given $y = f(x)$ where $x \leftarrow U_\lambda$, does the following:

1. Set $k = \lambda/\epsilon^2$.
2. For every $i \in [\lambda]$, do the following:

(a) Choose at random $r_{i,1}, \ldots, r_{i,k} \leftarrow \{0,1\}^\lambda$.

(b) For every $j \in [k]$ compute $b_{i,j} = \mathcal{A}(y, r_{i,j})$ and $b'_{i,j} = \mathcal{A}(y, r_{i,j} \oplus e_i)$, and let $x_{i,j} = b_{i,j} \oplus b'_{i,j}$.

(c) Let $x'_i = \mathsf{majority}\{x_{i,1} \ldots, x_{i,k}\}$

3. Output $x' = (x'_1, \ldots, x'_\lambda)$

Note that by our assumption for every $i \in [\lambda]$ and $j \in [k]$,

$$\Pr_{r_{i,j} \leftarrow \{0,1\}^\lambda}[x'_{i,j} = x_i] \geq$$

$$\Pr_{r_{i,j} \leftarrow \{0,1\}^\lambda}[\mathcal{A}(y, r_{i,j}) = x \cdot r_{i,j} \quad \wedge \quad \mathcal{A}(y, r_{i,j} \oplus e_i) = x \cdot (r_{i,j} \oplus e_i)] =$$

$$1 - \Pr_{r_{i,j} \leftarrow \{0,1\}^\lambda}[\mathcal{A}(y, r_{i,j}) \neq x \cdot r_{i,j} \quad \vee \quad \mathcal{A}(y, r_{i,j} \oplus e_i) \neq x \cdot (r_{i,j} \oplus e_i)] \geq$$

$$1 - \Pr_{r_{i,j} \leftarrow \{0,1\}^\lambda}[\mathcal{A}(y, r_{i,j}) \neq x \cdot r_{i,j}] - \Pr_{r_{i,j} \leftarrow \{0,1\}^\lambda}[\mathcal{A}(y, r_{i,j} \oplus e_i) \neq x \cdot (r_{i,j} \oplus e_i)] \geq$$

$$1 - 1/2 + 2\epsilon = 1/2 + 2\epsilon$$

Note that for every $i \in [\lambda]$ it holds that $x'_{i,1}, \ldots, x'_{i,k}$ are independent Boolean random variables such that $\Pr[x'_{i,j} = x_i] \geq 1/2 + 2\epsilon$. Therefore, by the Chernoff bound

$$\Pr[\mathsf{majority}\{x'_{i,1}, \ldots, x'_{i,k}\} \neq x_i] \leq 2^{-O(\epsilon^2 \cdot k)}$$

Theorem 3 (Chernoff bound)**.** *Let $X_1, X_2, \ldots, X_n$ be independent random variables taking values in $\{0,1\}$ (Bernoulli trials). Let*

$$X = \sum_{i=1}^{n} X_i, \qquad \mu = \mathbb{E}[X].$$

For any $0 < \delta < 1$:

$$\Pr[X \geq (1+\delta)\mu] \leq \exp\left(-\tfrac{\delta^2}{3}\mu\right),$$

and

$$\Pr[X \leq (1-\delta)\mu] \leq \exp\left(-\tfrac{\delta^2}{2}\mu\right).$$

Alternatively, the additive version asserts that for every $t > 0$

$$\Pr[X > \mu + t] \leq \exp\left(\tfrac{-2t^2}{n}\right)$$

and

$$\Pr[X < \mu - t] \leq \exp\left(\tfrac{-2t^2}{n}\right)$$

Thus, for every $i \in [\lambda]$

$$\Pr[\mathcal{B}(f(x)) = x' : \ x'_i \neq x_i] \leq 2^{-O(\lambda)},$$

which together with the union bound implies that

$$\Pr[\mathcal{B}(f(x)) \neq x] \leq \lambda \cdot 2^{-O(\lambda)} = \mathrm{negl}(\lambda).$$

Remark. Note that we could have chosen the same random $r_1, \ldots, r_k \leftarrow \{0,1\}^\lambda$ for all the coordinates $i \in [\lambda]$. The reason is that we do not need independence for the union bound.

General case: There exists a non-negligible $\epsilon = \epsilon(\lambda)$ such that for every $x \in \mathsf{GOOD}$, $\mathcal{A}$ predicts $P(x,r)$ with probability $1/2 + \epsilon$

Note that in the above analysis the reason that we needed a success probability of greater than $\frac{3}{4}$ is because, in order to get a non-negligible probability of guessing a single bit x_i we needed to combine *two* guesses for two bits $x \cdot r_{i,j}$ and $x \cdot (r_{i,j} \oplus e_i)$.

The Goldreich-Levin reduction starts with the following (ridiculous!) idea. Suppose that for each pair r and $r \oplus e_i$, we simply guess the bit $x \cdot r$ ourselves, and we only use the adversary $\mathcal{A}$ to guess the bit $x \cdot (r \oplus e_i)$. If we could somehow manage to always guess the bit $x \cdot r$ correctly then we could use the same argument as above since under this (ridiculous) assumption both bits $x \cdot r$ and $x \cdot (r \oplus e_i)$ are guessed correctly with probability at least $\frac{1}{2} + \epsilon$.

The problem is that we cannot guess $x \cdot r$ by ourselves with probability better than random. This is exactly the task that we needed $\mathcal{A}$ to do in the first place! But we can guess each of these bits at random and have a success probability $\frac{1}{2}$. This seems useless since in the above procedure, for each $i \in [\lambda]$ we needed to choose a total of k different values of $x \cdot r_j$, and if we guess each of them at random then we will guess all of them correctly only with probability 2^{-k}. But it turns out that this random guessing is actually really useful, due to the following clever idea: suppose that we happen to correctly guess $x \cdot s_1$ and $x \cdot s_2$. Then we can also guess correctly

$$x \cdot (s_1 \oplus s_2) = (x \cdot s_1) \oplus (x \cdot s_2)$$

More generally, suppose we guessed correctly

$$x \cdot r_1,\ x \cdot r_2,\ \ldots,\ x \cdot r_\ell$$

then we can use these ℓ bits to guess correctly

$$x \cdot (\oplus_{i \in I} r_i) = \oplus_{i \in I} (x \cdot r_i)$$

for every $I \subseteq [\ell]$. Since there are 2^ℓ such subsets we learned 2^ℓ inner products $x \cdot r_I$, where $r_I = \oplus_{i \in I} r_i$. Therefore, in order to learn k different inner products $x \cdot r_I$ we need to guess only $\ell = \log k$ many bits of the form $x \cdot r_j$! And we can guess all ℓ inner products correctly with probability $\frac{1}{k}$.

Still, all this may seem ridiculous since now these r_I's are not independent! For example, $r_{\{1,2\}} = r_1 \oplus r_2$. As a result we cannot use the Chernoff bound in the analysis, so it seems that we fixed one problem but created another. However, the problem we created is not a real problem. While indeed the r_I's are not all independent, they are *pairwise independent*, which turns out to be enough! Instead of using the Chernoff bound (which requires the random variables

to be independnet) we can rely on Chebyshev's inequality which is a weaker concentration bound but only requires the random variables to be pairwise independent.

Theorem 4 (Chebyshev's inequality)**.** *Let $Z_1, \ldots, Z_m$ be pairwise independent random variables obtaining values in $\{0,1\}$, such that for every $j \in [m]$ $\Pr[Z_j = 1] = p$. Then, for every $\delta > 0$,*

$$\Pr\left[\left|\frac{\sum_{j=1}^{m} Z_j}{m} - p\right| \geq \delta\right] \leq \frac{1}{4\delta^2 m}$$

So, the inverter $\mathcal{B}$, given $y = f(x)$, does the following:

1. Set $k = \lambda/\epsilon^2$.
2. For every $i \in [\lambda]$, do the following:
   (a) Set $\ell = \log k$
   (b) choose at random $r_1, \ldots, r_\ell \leftarrow \{0,1\}^\lambda$.
   (c) For every $j \in [\ell]$, choose at random $b_j \leftarrow \{0,1\}$.[1]
   (d) For every $J \subseteq [\ell]$, let
   $$r_J = \oplus_{j \in J} r_j \quad \text{and} \quad b_J = \oplus_{j \in J} b_j.$$
   (e) Similarly, for every $J \subseteq [\ell]$, let
   $$b'_{i,J} = \mathcal{A}(y, r_J \oplus e_i) \quad \text{and} \quad x_{i,J} = b_J \oplus b'_{i,J}.$$
   (f) Let $x'_i = \mathsf{majority}\{x_{i,J}\}_{J \subseteq [\ell]}$
3. Output $x' = (x'_1, \ldots, x'_\lambda)$

[1] b_j is a guess for $x \cdot r_j$.

Suppose that all our ℓ guesses where correct, which happens with probability $2^{-\ell} = \frac{1}{k}$. In this case, as above for every $i \in [\lambda]$ and $J \subseteq [\ell]$,

$$\begin{aligned}
&\Pr_{r_1,\ldots,r_\ell \leftarrow \{0,1\}^\lambda}[x'_{i,J} = x_i] \geq \\
&\Pr_{r_1,\ldots,r_\ell \leftarrow \{0,1\}^\lambda}[\mathcal{A}(y, (r_J + e_i)) = x \cdot (r_J \oplus e_i)] = \\
&1 - \Pr_{r_1,\ldots,r_\ell \leftarrow \{0,1\}^\lambda}[\mathcal{A}(y, r_J \oplus e_i) \neq x \cdot (r_J \oplus e_i)] \geq \\
&1/2 + \epsilon.
\end{aligned}$$

This seems great, except that now we cannot apply the Chernoff bound since the random variables $\{r_J\}_{J \subseteq [\ell]}$ are not independent! However, as mentioned above they are pairwise independent, and hence we can rely on Chebyshev's inequality to argue that for every $x \in \mathsf{GOOD}$ and every $i \in [\lambda]$,

$$\Pr[x'_i \neq x_i] \leq \frac{1}{4\epsilon^2 k}$$

In particular, setting $k = \lambda \cdot \epsilon^{-2}$, we can take a union bound, and guarantee that for every $x \in \mathsf{GOOD}$,

$$\Pr[\mathcal{B}(f(x) = x] \geq \frac{1}{4},$$

contradicting the fact that f is a one-way function.

□

References

[1] Oded Goldreich and Leonid A. Levin. A hard-core predicate for all one-way functions. In *Proceedings of the 21st Annual ACM Symposium on Theory of Computing (STOC)*, pages 25–32, Seattle, Washington, USA, 1989.